JOURNAL OF MATERIALS SCIENCE 18 (1983) 2354–2360

The potential drop across an imperfect diffusion bond

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In the non-destructive evaluation of diffusion bonds, a possible technique is to measure the potential drop for a given current flow. Using evidence from optical and acoustic microscopy, the geometrical details of an incomplete diffusion bond are represented by an idealized mathematical approximation. This is solved exactly using conformal mapping by a Schwarz transformation. The result for the change in resistance as a function of bond development is given as an analytic expression.

1. Statement of problem

In a companion paper [1], various methods for the non-destructive assessment of diffusion bonds are compared and evaluated. In all of these an attempt is made to determine the true bonded area, since this is the dominant factor influencing the strength of the bond. The potential drop technique is already well established for measurements of fatigue crack growth [2, 3] and it has now been tried for monitoring diffusion bonds [1], though further refinement of the experimental technique is still required.

In calculating the change in resistance due to incomplete bonding, the problem is formulated as follows. The surfaces to be bonded are ground, and then brought together with the grinding directions parallel. They are then pressed together at a pressure less than the yield stress and at a temperature less than the yield point, so that bond growth proceeds by diffusional processes [4]. In any bond where the diffusional processes are incomplete, a series of voids will remain corresponding to troughs left by the grinding process. These voids tend to be rather thin in the direction normal to the bonded interface and long in the direction parallel to the grinding marks. Perpendicular to the grinding marks, if the surface undulations can be given a characteristic wavelength, the voids are spaced at intervals corresponding to the wavelength, with a width of some fraction of the wavelength. In a well-prepared specimen the surface roughness is greater than the lack of flatness, so that the voids are of similar sizes and shapes over the whole of the bonded area. Some evidence from optical and acoustic microscopy to support these observations is presented in the companion paper [1].

In order to make the problem tractable mathematically, the following approximations are introduced. First it is assumed that the voids are of length equal to the thickness of the specimen; this reduces the problem to two dimensions. Second, it is assumed that the voids are of infinitessimal thickness; this simplifies the Schwarz transformation. They also have zero conductivity, so that no current flows through them. Third, it is assumed that the voids are of regular width and spacing, so that the problem is periodic. This description is illustrated in Fig. 1. Since no current flows across the broken lines shown, the problem reduces to finding the change in resistance of one cell, and from this the resistance of the bond can be deduced. The final assumption is that the specimen is of infinite length perpendicular to the bond. For simplicity, a bond of rectangular cross-section will be described, though the result will be valid for a bond of any cross-section.

2. Analysis

The problem is tackled by employing a conformal transformation [5]. This transforms the complicated boundary conditions of the given geometry,

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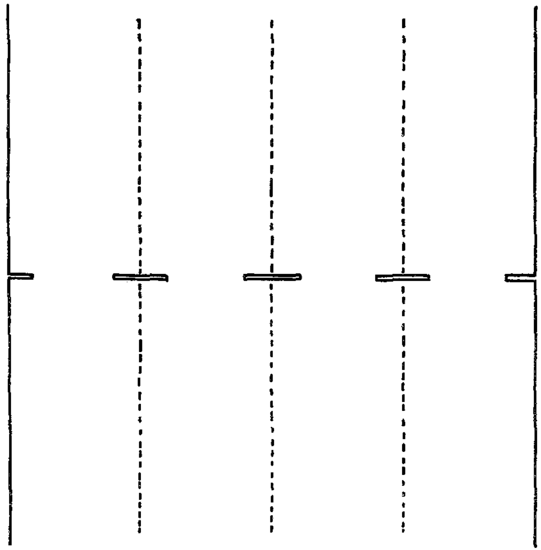

Figure 1 The geometry of the description of a diffusion bond used for the calculations. It consists of a number of identical cells in parallel, as indicated by the broken lines.

taken in a complex plane, to another geometry for which the boundary conditions are simple. Since Laplace's equation is invariant under the transformation, it may be applied to the transformed geometry and the solution then inverse-transformed to give the required result.

2.1. The Schwarz transformation

The required transformation is illustrated in Fig. 2, where the points corresponding to A, B, C ... in the z-plane are A_1, B_1, C_1 ... in the z_1-plane. For z contained between the two boundaries ($|\text{Re}\, z| \leqslant h$), the limit $\text{Im}\, z \to -\infty$ will map onto the origin of the z_1 plane, while the limit $\text{Im}\, z \to +\infty$ will map onto a semicircle of infinite radius in the upper half ($y_1 \geqslant 0$) of the z_1 plane. The left-hand boundary of the material maps onto the negative x_1 axis and the right-hand boundary onto the positive x_1 axis.

With bends of $\pi/2$, 2π, $\pi/2$, 0, $\pi/2$, 2π, $\pi/2$ at $z_1 = -a$, -1, $-b$, 0, b, 1, a, respectively, the Schwarz transformation between the two planes becomes

$$\frac{dz}{dz_1} = c\frac{z_1^2 - 1}{z_1[(a^2 - z_1^2)(z_1^2 - b^2)]^{1/2}} = f(z_1) \quad (1)$$

2.1.1. Interpretation of the square root

The terms in the denominator of Equation 1 are illustrated in Fig. 3. Since the square root is multivalued we make a precise interpretation of it from the start by

$$\{-R_1R_2R_3R_4 \exp[i(\phi_1 + \phi_2 + \phi_3 + \phi_4)]\}^{1/2} = (R_1R_2R_3R_4)^{1/2} \times \exp[i(\pi + \phi_1 + \phi_2 + \phi_3 + \phi_4)/2] \quad (2)$$

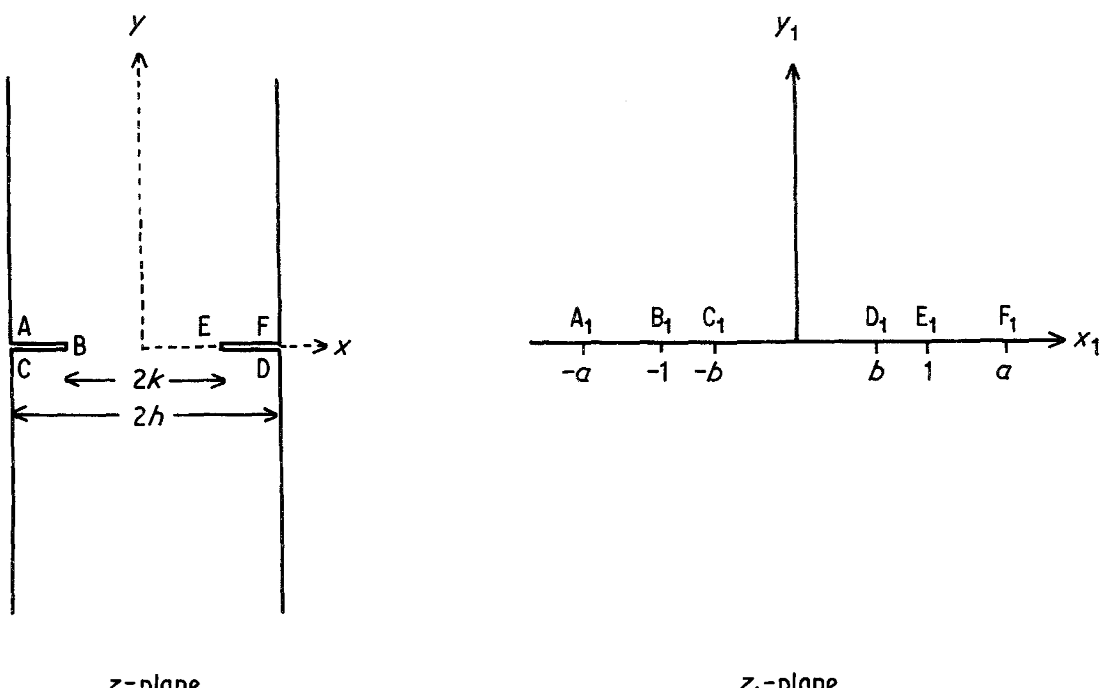

Figure 2 The two complex planes z and z_1. A cell of the diffusion bond in z is mapped to z_1 by the Schwarz transformation of Equation 1.

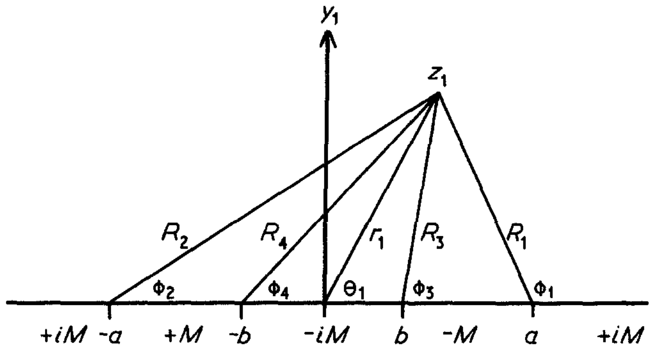

Figure 3 The interpretation of the square root in the denominator of Equation 1. For z_1 along the real axis, the square root takes the values indicated, where M is the modulus.

where the square root on the right is always taken as positive. Thus for z_1 along the real axis, the square root takes the values indicated in Fig. 3.

2.1.2. Simple relations between constants

Simple relations between the constants a, b, c, h can be obtained from the requirements

$$\int_{h+i\infty}^{-h+i\infty} dz = -2h$$
$$= \lim_{R\to\infty} \int_{r_1=R,\theta_1=0}^{r_1=R,\theta_1=\pi} f(z_1)\, dz_1.$$

and

$$\int_{h-i\infty}^{-h-i\infty} dz = -2h$$
$$= \lim_{R\to 0} \int_{r_1=R,\theta_1=0}^{r_1=R,\theta_1=\pi} f(z_1)\, dz_1.$$

The two limits of the integral on the right may be evaluated, after substituting from Equation 1, in an entirely equivalent manner to that used by Smythe [5] for the problem of a strip with an abrupt width change. This gives the result

$$c = -\frac{2h}{\pi}, \quad ab = 1. \tag{3}$$

2.2. Transformation of the integral

The Schwarz transformation can now be expressed in the form

$$z = -\frac{2h}{\pi}\int \frac{z_1^2 - 1}{z_1[(a^2 - z_1^2)(z_1^2 - 1/a^2)]^{1/2}}\, dz_1 + \text{constant}. \tag{4}$$

If we consider a change of variable to

$$s = [(a^2 - z_1^2)(z_1^2 - 1/a^2)]^{1/2}/(z_1^2 + 1) \tag{5}$$

where the square root is interpreted as in Equation 2, then it is readily shown after some straightforward algebra that Equation 4 transforms to

$$z = \frac{2h}{\pi}\int \frac{ds}{s^2+1} + \text{constant}$$
$$= \frac{2h}{\pi} \tan^{-1}s + \text{constant}. \tag{6}$$

Now, $\tan^{-1}s$ has a cut in the s-plane as indicated in Fig. 4, and in performing the integration in Equation 6 this cut must not be crossed. But under the transformation of Equation 5, it can be seen that the y_1-axis for positive values of y_1 is mapped onto the cut of $\tan^{-1}s$. Particular values of this mapping are

$$y_1 \to 0^+, s \to -i1^+;\ y_1 \to 1^-, s \to -i\infty$$
$$y_1 \to \infty,\ s \to +i1^+;\ y_1 \to 1^+, s \to +i\infty$$

where the superscripts indicate the direction from which a limit is approached.

It is therefore necessary to carry out the integration in Equation 6 and the determination of the constant of integration independently for

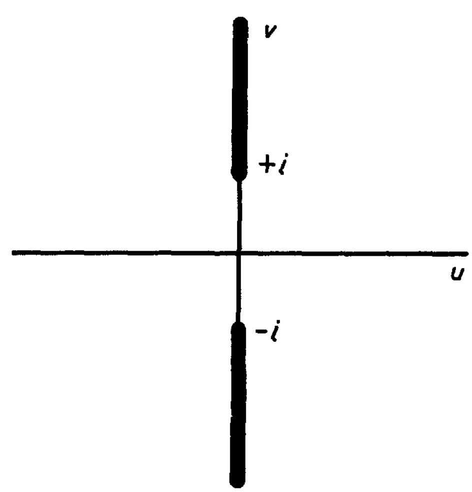

Figure 4 The cut in $\tan^{-1}s$, shown in the s-plane.

positive and negative values of $\text{Re}(z_1)$, and thus for the right and left hand halves of our original geometry, in order that the cut remain uncrossed and Equation 6 remain valid. Eventually the two solutions will have to be matched.

2.2.1. Evaluation of the integral

Consider first the case of $\text{Re}(z_1) > 0$. The Schwarz transformation was designed to have $z = h$ when $z_1 = a$, which also corresponds, by Equation 5, to $s = 0$. The constant in Equation 6 is thus evaluated as h. On the other hand, for $\text{Re}(z_1) < 0$, we require $z = -h$ when $z_1 = -a$, when again $s = 0$. The constant of integration then becomes $-h$. The transformation is thus

$$z = \frac{2h}{\pi} \tan^{-1} s \pm h \qquad (7)$$

with the sign of h corresponding to the sign of $\text{Re}(z_1)$.

The constant a is as yet undetermined. This can be fixed by requiring that $z = k$ when $z_1 = 1$. Substituting Equation 5 into Equation 6 gives

$$k = \frac{2h}{\pi} \tan^{-1} \left(-\frac{a^2 - 1}{2a} \right) + h$$

which may be expressed alternatively in the form

$$a = \tan \alpha + \sec \alpha, \quad \alpha = \left(1 - \frac{k}{h}\right) \frac{\pi}{2}. \qquad (8)$$

The same result obtains using the requirement that $z = -k$ when $z_1 = -1$, and, of course, using Equation 7 in the appropriate form.

2.2.2. Matching the solutions

The Schwarz transformation is now fully determined by Equations 7, 5 and 8. However, we expect that the transformation should be continuous across the positive y_1-axis of the z_1-plane. Thus the change in sign of the constant of integration in Equation 7, as z_1 crosses the positive y_1-axis, must be compensated by the discontinuous change in $\tan^{-1} s$ as s crosses the cut in the s-plane. That this is indeed true may be shown as follows.

With s specified as $u + iv$, the values of $\tan^{-1} s$ near the cut may be obtained from

$$\lim_{u \to 0^\pm} \tan^{-1}(s) = \lim_{u \to 0^\pm} \frac{1}{2i} \ln \left(\frac{1 + is}{1 - is} \right)$$

$$= \lim_{u \to 0^\pm} \frac{1}{2i} \ln \left(\frac{1 - u^2 - v^2 + i2u}{(1 + v)^2 + u^2} \right). \qquad (9)$$

Then, writing the argument of the logarithm as $t = R\exp(i\phi)$,

$$\text{as } u \to 0^\pm, \qquad \text{Re}(t) \to \frac{1 - v^2}{(1 + v)^2} = \frac{1 - v}{1 + v}$$

$$\text{and } \text{Im}(t) \to 0^\pm. \qquad (10)$$

Thus, for the uncut section of the imaginary axis ($|v| < 1$), $\text{Re}(t) > 0$, and so as $\text{Im}(t) \to 0^\pm$ $\phi \to 0$. We may then write

$$\lim_{u \to 0^\pm} \tan^{-1}(s) = \frac{1}{2i} \ln \left(\frac{1 - v}{1 + v} \right), |v| < 1. \qquad (11)$$

However, for the cut section of the imaginary axis ($|v| > 1$), Equation 10 shows that $\text{Re}(t) < 0$ and thus as $\text{Im}(t) \to 0^+$, then $\phi \to \pi$, while as $\text{Im}(t) \to 0^-$, then $\phi \to -\pi$, since the cut in the logarithm consistent with the use of Equation 9 is along the negative real axis. Thus,

$$\lim_{u \to 0^\pm} \tan^{-1}(s) = \pm \frac{\pi}{2} + \frac{1}{2i} \ln \left| \frac{1 - v}{1 + v} \right|. \qquad (12)$$

Furthermore, one may show from Equation 5, that

$$\text{as } z_1 \to iy_1 + 0^\pm \quad \text{then } s \to iv - 0^\pm$$

and we have already seen that the positive imaginary axis in the z_1-plane is mapped onto the cut in the s-plane by Equation 5. Thus taking the limit of Equation 7 as z_1 approaches the positive imaginary axis from either side,

$$\lim_{\substack{x_1 \to 0^\pm \\ (0 < y_1 < \infty)}} z = \frac{2h}{\pi} \lim_{\substack{u \to 0^\mp \\ (|v| > 1)}} \tan^{-1} s \pm h$$

$$= \frac{2h}{\pi} \left[\mp \frac{\pi}{2} + \frac{1}{2i} \ln \left| \frac{1 - v}{1 + v} \right| \right] \pm h$$

$$= -i \frac{h}{\pi} \ln \left| \frac{1 - v}{1 + v} \right|. \qquad (13)$$

Thus the solutions reach the same limiting value from each side of the cut.

2.3. Determination of the differential resistance

The method of determining the differential resistance is similar to that used by Smythe [5] for the differential resistance caused by an abrupt width change. That is, we require a solution of Laplace's equation which gives infinite positive and negative potentials in the regions of the z_1 plane corresponding to $z \to i\infty$ ($z_1 \to i\infty$) and $z \to -i\infty$ ($z_1 \to 0$), and which has no gradient across the real z_1 axis. If we write

$$W = U + iV$$

where U is the potential function, then such a solution is

$$W = \ln z_1 \quad \text{or} \quad z_1 = e^W \tag{14}$$

Substituting this into Equation 5, and considering only points on the positive y_1-axis, where V is equal to $\pi/2$, gives

$$s = iv = \frac{-i|\{(a^2 + e^{2U})[(1/a^2) + e^{2U}]\}^{1/2}|}{1 - e^{2U}} \tag{15}$$

again interpreting the square root by Equation 2.

Then, in the limit $U = U_1 \to \infty$,

$$v = [e^{2U_1} + \tfrac{1}{2}(a^2 + 1/a^2)]/[e^{2U_1} - 1] \tag{16}$$

which from Equation 13 is obtained for a value of y given by

$$y_+ = -\frac{h}{\pi} \ln \left|\frac{1-v}{1+v}\right|$$

$$= \frac{2hU_1}{\pi} - \frac{2h}{\pi} \ln\left(\frac{a^2+1}{2a}\right). \tag{17}$$

On the other hand, in the limit $U = U_2 \to -\infty$,

$$v = [1 + e^{2U_2}(a^2 + 1/a^2)/2]/[-1 + e^{2U_2}] \tag{18}$$

which is obtained for a value of y given by

$$y_- = -\frac{h}{\pi} \ln \left|\frac{1-v}{1+v}\right|$$

$$= \frac{2hU_2}{\pi} + \frac{2h}{\pi} \ln\left(\frac{a^2+1}{2a}\right). \tag{19}$$

Combining Equations 17 and 19,

$$U_1 - U_2 = \frac{\pi}{2h}(y_+ - y_-) + 2\ln\left(\frac{a^2+1}{2a}\right) \tag{20}$$

If the two-dimensional resistivity (i.e. the resistance between opposite sides of a unit square) of the conducting material is s', then Ohm's law may be written as [5]

$$R = s'\frac{|U_2 - U_1|}{|V_2 - V_1|}$$

where V_1 and V_2 are the bounding lines of force, in this case 0 and π, respectively. This gives

$$R = \frac{s'}{2h}(y_+ - y_-) + \frac{2s'}{\pi} \ln\left(\frac{a^2+1}{2a}\right). \tag{21}$$

The first term on the right is the resistance of a uniform strip. Therefore the change in resistance is

$$\Delta R = \frac{2s'}{\pi} \ln\left(\frac{a^2+1}{2a}\right). \tag{22}$$

This is essentially the result we have been seeking, and it may be immediately applied to the diffusion bond of Fig. 1. If the gross bond area is A and the resistivity of the material is ρ, then the change in resistance due to imperfect bonding, compared with the resistance of the same length of uniform material, measured between planes parallel to the bond and a distance from it large compared with h, is

$$\Delta R = \frac{4h\rho}{\pi A} \ln\left(\frac{a^2+1}{2a}\right) \tag{23}$$

with

$$a = \tan\alpha + \sec\alpha$$

where

$$\alpha = \left(1 - \frac{k}{h}\right)\frac{\pi}{2}.$$

This result is plotted in dimensionless form in Fig. 5.

3. Discussion: the nearly perfect bond

The practical implications of this calculation are discussed in the companion paper [1], where the result is also plotted semi-logarithmically. We consider here the case of the nearly perfect bond, i.e. $1 - k/h \ll 1$.

First, we find the slope of ΔR at the limit $k = h$

$$\alpha = 0$$

$$a = 1$$

$$\frac{h\,\mathrm{d}\alpha}{\mathrm{d}k} = -\frac{\pi}{2}$$

$$\frac{\mathrm{d}a}{\mathrm{d}\alpha} = \sec^2\alpha - \sec\alpha\tan\alpha$$

$$= 1$$

$$\frac{\mathrm{d}}{\mathrm{d}a}\left[\ln\left(\frac{a^2+1}{2a}\right)\right] = \frac{a^2-1}{a(a^2+1)}$$

$$= 0$$

$$\therefore\ h\,\frac{\mathrm{d}\Delta R}{\mathrm{d}k} = 0.$$

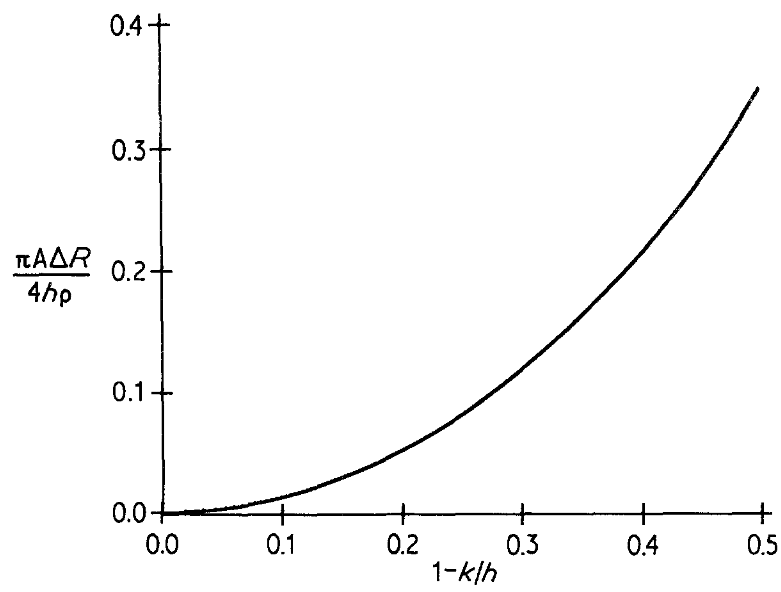

Figure 5 The change in resistance ΔR of Equation 23, plotted in dimensionless form.

This describes the fact that the slope in Fig. 5 is zero when $1 - k/h = 0$. It means that the p.d. method rapidly loses sensitivity as a bond becomes more nearly perfect. This may not be as serious a practical limitation as it sounds, since bonds corresponding to $k/h = 0.8$ can have almost 100% strength [1].

As $k/h \to 1$, the approximation that the voids are planar becomes less valid. As a better approximation the voids might be considered to be of rectangular section, as shown in Fig. 6, with thickness $2l$. Following an analysis similar to Section 2, the mapping is defined by

$$\frac{dz}{dz_1} = C_1 \frac{(z_1^2 - a^2)^{1/2}(z_1^2 - b^2)^{1/2}}{z_1(z_1^2 - 1)^{1/2}(z_1^2 - c^2)^{1/2}} \tag{24}$$

where

$$C_1 = \pm \frac{2ik}{\pi}$$

$$\frac{ab}{c} = 1$$

$$-2il = -\frac{C_1}{2} \int_{a^2}^{b^2} \frac{(s - a^2)^{1/2}(s - b^2)^{1/2}}{s(s-1)^{1/2}(s - c^2)^{1/2}} \, ds$$

$$h - k = -\frac{C_1}{2} \int_{b^2}^{c^2} \frac{(s - a^2)^{1/2}(s - b^2)^{1/2}}{s(s-1)^{1/2}(s - c^2)^{1/2}} \, ds$$

$$s = z_1^* z_1$$

The solution of this mapping is left as an excercise for the interested reader! It will follow lines similar to those of Section 2, though it will be more complicated. For $h - k \gg l$ there will be little difference in the results; the chief new feature will be that for constant l the slope of ΔR will be non-zero at $k = h$. However, it would remain a matter of discussion to what extent this would be

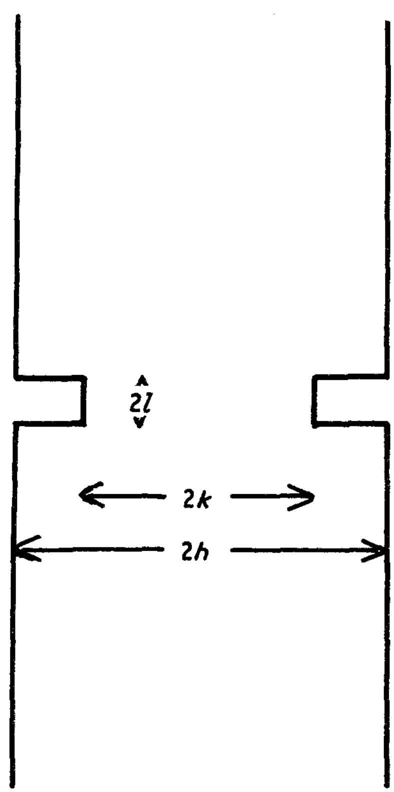

Figure 6 An alternative idealization of a diffusion bond in which the voids have finite thickness $2l$. The Schwarz transformation for this geometry is given in Equation 24.

a better mathematical description of a nearly perfect bond, and for most practical purposes the result of Section 2 is commended.

Acknowledgements

We are grateful for support from AERE for GADB. We wish to express our thanks to Dr B. Derby and Dr E. R. Wallach for bringing this problem to our attention, to Dr N. Louat for early discussions and to Dr M. G. Somekh for plotting Fig. 5.

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Received 8 November and accepted 20 December 1982